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Satellite Formation Design and Optimal Stationkeeping Considering Nonlinearity and Eccentricity

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I. Introduction

FORMATION flying has received much attention in recent years because of the possible advantages of replacing a single, complex satellite with a cluster of smaller ones. Flying a formation of satellites offers improved flexibility and redundancy and the ability to construct much large virtual sensors than can be flown on a single, monolithic satellite. Several missions have identified formation flying as an enabling technology for increasing the science return and reducing the total mission costs [1,2].

A critical problem in the formation-flying application is the formation design. The purpose of formation design is twofold. First, various flying missions have different requirements for formation design, such as the Aperture Synthesis Radar mission, the Earth Observer-1 mission, the Laser Interferometer Space Antenna mission, etc. The primary purpose of formation design is searching the proper formation array that fulfills the formation mission requirement. The scientific or other need of the mission is the main constraint for the formation design and can be defined as the mission constraint for the formation design. Second, formation-flying satellites operate on orbits with long time spans, in general, from several months to several years, and frequent thruster firings to keeping formation will consume so much propellant and cannot be accepted by many considering missions. And so, it is more important to design nature periodic relative motion orbits and avoid the secular drift. This can be defined as the orbit constraint for formation design.

The problem of satellite formation design has been studied by many researchers and many advances have been made. Sabol et al. [3] used the Hill's equations to design formation and proposed four special formations for various formation-flying missions. Hill's equations are the constant coefficient differential equations and have simple analytical solutions with an in-track secular term in the solutions. The drift can be avoided by a proper initialization. These simple analytical expressions permit the use of intuitive methods to design formations. However, Hill's equations can only design valid formations for circular reference orbits and linearized relative motion. Carter et al. [4–6] studied the relative motion of two vehicles in nearby elliptical orbits and presented an analytical solution under assumptions of linearization and without perturbations, and developed an initialization procedure to obtain initial conditions for the formation satellites period relative motion at reference orbit perigee. But, the solutions of Carter's have a definite integral and are complex in form and cannot be conveniently applied in formation design. Using orbital element differences and Cartesian coordinates, respectively, Lane and Axelrad [7] and Xing et al. [8] derived the simple periodic analytic solutions in eccentric orbits under assumptions of linearization. These works investigated the linearized problem of relative motion of formation flying. Considering nonlinearity and eccentricity, Gurfil [9] and Xing et al. [10,11] independently proposed generalized periodic relative motion conditions (GPRMC) for formation flying on arbitrary Keplerian elliptic orbits.

This paper focuses on the formation design for nonlinear relative motion and eccentric reference orbits under GPRMC. Because there are some linearized errors for formation design using Hill's equations and solutions of Xing's [8], a new method is proposed to design formation considering nonlinearity and eccentricity which use GPRMC to correct the initial conditions derived from Hill's equations or Xing's solutions and get the periodic relative motion orbits for formation flying. This approach is attractive for two reasons. First, the process of formation design uses the GPRMC and removes the secular drift of relative motion orbit designed by Hill's equations or Xing's solutions and avoids frequent thruster firings to keep formation. More important, the approach adequately uses the previous research results which were derived from Hill's equations and could use the intuitive methods to design formations for various formation-flying missions.

It is assumed that the formation designed by Hill's equations fulfilled the requirement of the reality formation-flying mission. Considering nonlinearity and eccentricity, the designed formations were not perfectly consistent with Hill's. The remainder of this paper developed an optimal impulse formation-keeping maneuver based on the orbit constraint (GPRMC) and the mission constraint (Hill's solutions). We used the orbit constraint and the mission constraint to keep periodic relative motion and fulfill the formation-flying mission requirement, respectively.

II. Generalized Periodic Relative Motion Conditions

Consider two spacecraft orbiting a common point-mass central body. One spacecraft, flying on a given reference orbit, is termed the leader and the other is referred to as the follower.

Shown in Fig. 1, we adopted the Hill coordinate system O, centered at the formation spacecraft. The unit vector is directed from the spacecraft radially outward, $\hat{z}$ is normal to the orbital plane, and $\hat{y}$ completes the right-handed coordinate system. The subscripts l and f express the leader and follower, respectively.

In the O_l frame, the leader's velocity vector is

$$\dot{\mathbf{r}}_l = \left\{\omega_l r_l \frac{e_l \sin\theta_l}{1 + e_l \cos\theta_l},\ \omega_l r_l,\ 0\right\}^T \tag{1}$$

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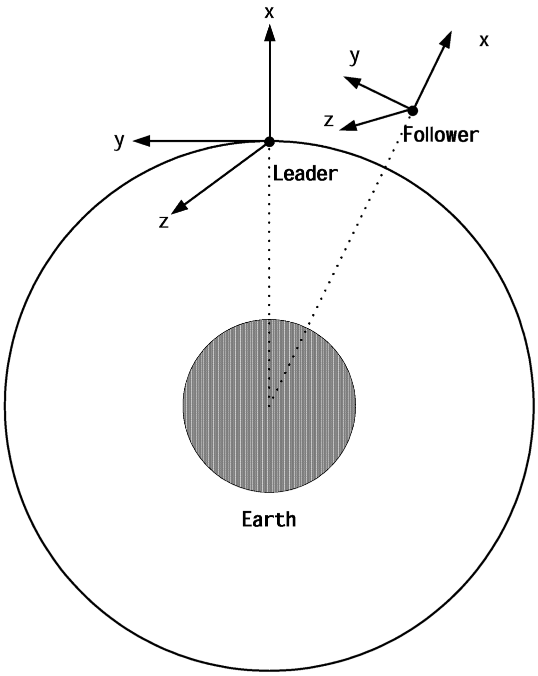

Fig. 1 Hill coordinate systems centered at the leader and follower, respectively.

and the follower's velocity vector with respect to the leader is

$$\dot{\mathbf{r}}_f = \left\{ \dot{x} - \omega_l y + \omega_l r_l \frac{e_l \sin\theta_l}{1 + e_l \cos\theta_l},\ \dot{y} + \omega_l (r_l + x_l),\ \dot{z} \right\}^T \tag{2}$$

The total specific energy of the leader spacecraft comprising the kinetic and potential energies is

$$E_l = \frac{1}{2}\left(\omega_l r_l \frac{e_l \sin\theta_l}{1 + e_l \cos\theta_l}\right)^2 + \frac{1}{2}(\omega_l r_l)^2 - \frac{\mu}{r_l} = -\frac{\mu}{a_l} \tag{3}$$

The total energy of the follower is

$$E_f = \frac{1}{2}\left(\dot{x} - \omega_l y + \omega_l r_l \frac{e_l \sin\theta_l}{1 + e_l \cos\theta_l}\right)^2 + \frac{1}{2}[\dot{y} + \omega_l (r_l + x)]^2 + \frac{1}{2}\dot{z}^2 - \frac{\mu}{\sqrt{(r_l + x)^2 + y^2 + z^2}} = -\frac{\mu}{a_f} \tag{4}$$

The energy difference between the leader and follower can be written as

$$\Delta E = E_f - E_l = \frac{\mu}{2a_l} - \frac{\mu}{2a_f} \tag{5}$$

Under the influence of the point-mass central body only, whether the reference orbits are circular or elliptic, the periodic relative motion among the formation spacecrafts requires period matching. At the same time, the period-matching condition guarantees the periodic relative motion among the formation spacecrafts. And so the necessary and sufficient condition for the periodic relative motion is

$$\Delta E = 0 \tag{6}$$

Because the energy of spacecraft is conserved in two-body gravitation, Eq. (6) is satisfied for all time. Therefore Eq. (6) constitutes the generalized periodic relative motion condition at any point along the follower's orbit. Substituting Eqs. (1–4) into Eq. (6), the periodic relative motion can be written as

$$\frac{1}{2}(\dot{x} - \omega_l y)^2 + \frac{1}{2}(\dot{y} + \omega_l x)^2 + \frac{1}{2}\dot{z}^2 + \omega r_l (\dot{y} + \omega_l x) + \omega r_l \frac{e_l \sin\theta_l}{1 + e_l \cos\theta_l}(\dot{x} - \omega_l y) + \left[\frac{\mu}{r_l} - \frac{\mu}{\sqrt{(r_l + x)^2 + y^2 + z^2}}\right] = 0 \tag{7}$$

Unlike Hill's initial condition, in deriving Eq. (7), the linear and circular assumption cannot be made. And so Eq. (7) is satisfied for nonlinear relative motion and arbitrary Keplerian elliptic reference orbits.

When the relative distance between the formation spacecraft is much smaller than the semimajor axis of the leader, from Eq. (6), the linearized periodic relative motion condition is given as

$$(\dot{y} + \omega_l x) + \frac{e_l \sin\theta_l}{1 + e_l \cos\theta_l}(\dot{x} - \omega_l y) = -\frac{\omega_l}{(1 + e_l \cos\theta_l)} x \tag{8}$$

Especially, when $e_l = 0$ in Eq. (8), we have

$$\dot{y} = -2\omega_l x \tag{9}$$

This condition is the same as the initial condition of Hill's equations [3].

Similarly, when $e_l \neq 0$ and $\theta_l = 0$ in Eq. (8), we have

$$\dot{y}_0 = -\frac{n_l (e_l + 2)}{(1 + e_l)^{1/2}(1 - e_l)^{3/2}} x_0 \tag{10}$$

Using θ as the free variable, Eq. (10) can be transformed using the relationships

$$\frac{\mathrm{d}y}{\mathrm{d}\theta} = \frac{\mathrm{d}y}{\mathrm{d}t}\frac{\mathrm{d}t}{\mathrm{d}\theta} = \frac{(1 - e_l)^{\frac{3}{2}}}{n_l (1 + e_l)^{\frac{1}{2}}} \dot{y} \tag{11}$$

With this transformation, Eq. (10) can be written as

$$\left.\frac{\mathrm{d}y}{\mathrm{d}\theta}\right|_{\theta=0} = -\frac{e_l + 2}{e_l + 1} x_{\theta=0} \tag{12}$$

Equations (10) and (12) are the initial conditions of periodic motion for formation flying on arbitrary elliptic orbits. Those are the same as [6].

From the above results, we can conclude that Eq. (7) is GPRMC considering nonlinearity and eccentricity and the previous results (in [3,6]) are only the specific cases under the linearization assumption. Unlike the previous methods to find the initial conditions (in [3,6]), which derived from solving the relative motion differential equations, Eq. (7) was derived by some algebraic manipulation and the process is simpler than the previous methods.

III. Formation Design Considering Nonlinearity and Eccentricity

The relative motion dynamics for an eccentric reference orbit is governed by the following nonlinear differential equations [12]:

$$\begin{aligned} \ddot{x} - 2\dot{\theta}_l \dot{y} - \dot{\theta}_l^2 x - \ddot{\theta}_l y &= \frac{\mu}{r_l^2} - \frac{\mu (r_l + x)}{[(r_l + x)^2 + y^2 + z^2]^{3/2}} \\ \ddot{y} + 2\dot{\theta}_l \dot{x} + \ddot{\theta}_l x - \dot{\theta}_l^2 y &= -\frac{\mu y}{[(r_l + x)^2 + y^2 + z^2]^{3/2}} \\ \ddot{z} &= -\frac{\mu z}{[(r_l + x)^2 + y^2 + z^2]^{3/2}} \end{aligned} \tag{13}$$

From Eqs. (9) and (10) we know that the initial conditions of Hill's and Xing's are only the specific cases under the linearization assumption and there are the secular drift terms in the relative motion if we use those conditions to design formations. This is illustrated in example 1.

Example 1: Considering a leader satellite on the circular orbit with semimajor axis 8000 km and the formation is the circular projection relative orbits [3] given by the circular radius ρ of 10 km. Hill's initial conditions are

$$x_0 = 0, \quad y_0 = \rho, \quad z_0 = 0, \quad \dot{x}_0 = \omega\rho/2, \quad \dot{y}_0 = 0, \quad \dot{z}_0 = \omega\rho \tag{14}$$

Integrating Eq. (13) with Hill's initial conditions, the relative motions in the configure space are shown in Fig. 2. Using Hill's initial conditions to design the formation, the GPRMC cannot be satisfied and the periods of formation spacecraft are not matching, so there are the secular drift terms in the relative motion. The secular terms lead to unbounded relative motion and it must be avoided by the proper methods.

We can use the GPRMC to correct initial conditions of Hill's or Xing's to match the periods of formation spacecrafts and avoid the secular growth in the relative motion. From Eq. (7), it is known that the valid method to satisfy the zero secular growth requirement is justifying x or $\dot{y}$. We selected $\dot{y}$ in this example.

From Eqs. (7) and (14), the corrected value of $\dot{y}$ is 0.0124 m/s. Integrating Eq. (13) with the corrected initial conditions of Hill's, the relative motions in the configure space are shown in Fig. 3. It shows that there are no secular drift terms and the relative motion is bounded and periodic.

When integrating the nonlinear dynamic Eq. (13) with GPRMC, the projected circular radius is not equal to the radius designed by Hill's equations. The projected circular radius error between Hill's and the nonlinear dynamic equation is shown in Fig. 4. It shows that the radius error is changing periodically and the maximum error is 14 m, only 0.14% of the radius ρ. We assume the projected circular formation designed by Hill's equations is the nominal formation which fulfills the requirement of the reality formation-flying mission. If the designed allowable error of the radius is bigger than 0.14%, we do not need the thruster firings to keep the formation, whereas we need control to keep the formation.

IV. Impulsive Formation Keeping for Periodic Relative Motion and Mission Requirement

Equation (7) constitutes a necessary and sufficient condition for periodic relative motion. However, in engineering practice, because of initialization errors and disturbing force effect, this constraint cannot be satisfied exactly, which causes the secular drift in relative motion. To compensate for such errors and reduce a disturbing force effect, the follower spacecraft must perform a formation-keeping maneuver. At the same time, an initialization error and disturbing force also perturb the nominal formation which fulfills the requirement of the reality formation-flying mission, and the maneuver must be performed to keep nominal formation. A single-impulse formation-keeping maneuver strategy was developed for periodic relative motion while consuming minimum fuel, respectively.

A. Formation Keeping for Periodic Relative Motion

In the O_f frame, the GPRMC can be written as

$$\Delta E = \left(\omega_f r_f \frac{e_f \sin\theta_f}{1 + e_f \cos\theta_f} + \Delta v_x\right)^2 + (\omega_f r_f + \Delta v_y)^2 + (\Delta v_z)^2 + \frac{\mu}{a_l} - \frac{\mu}{r_f} = 0 \tag{15}$$

where

$$r_f = \frac{(a_l + \Delta a)(1 - e_f^2)}{1 + e_f \cos\theta_f} \tag{16}$$

In this case, the impulsive maneuver parameter optimization problems can be stated as follows:

Find an optimal impulsive maneuver $\Delta\mathbf{v}$, satisfying

$$\Delta v = \min_{\Delta v} \|\Delta\mathbf{v}\|^2 \qquad \text{s.t. } \Delta E = 0 \tag{17}$$

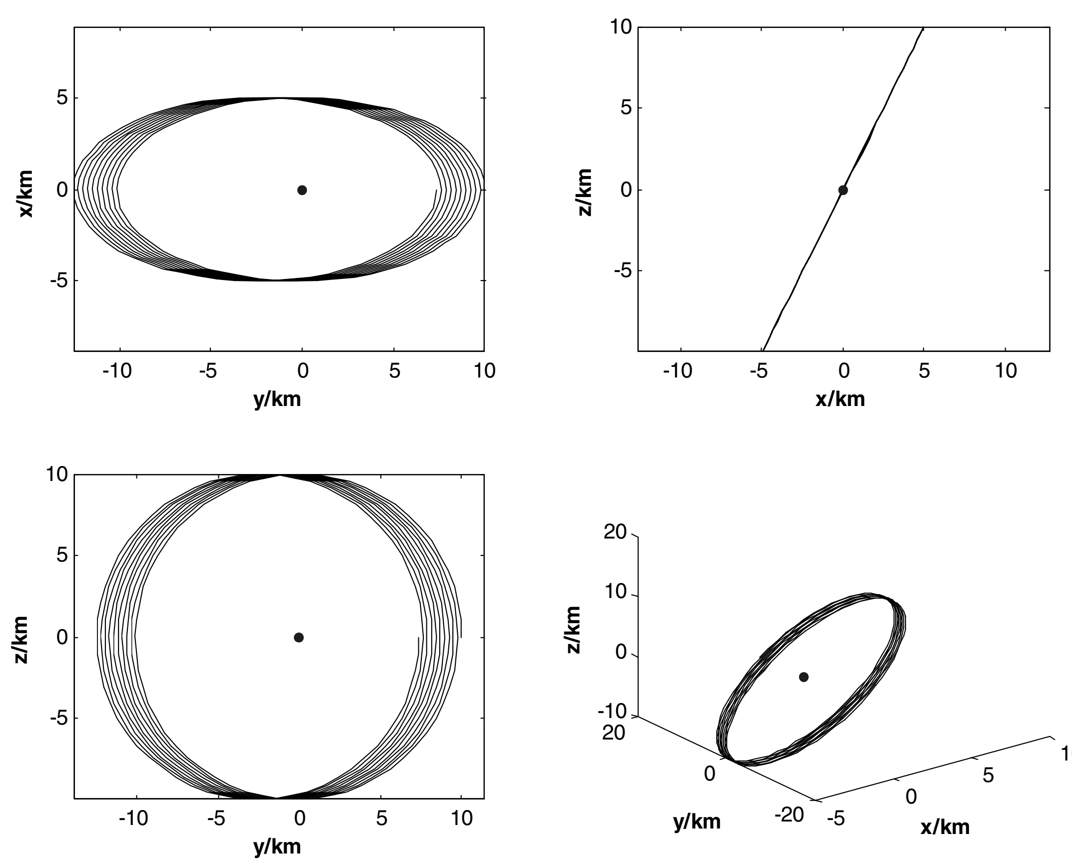

Fig. 2 Relative motions in the configure space with Hill's initial conditions.

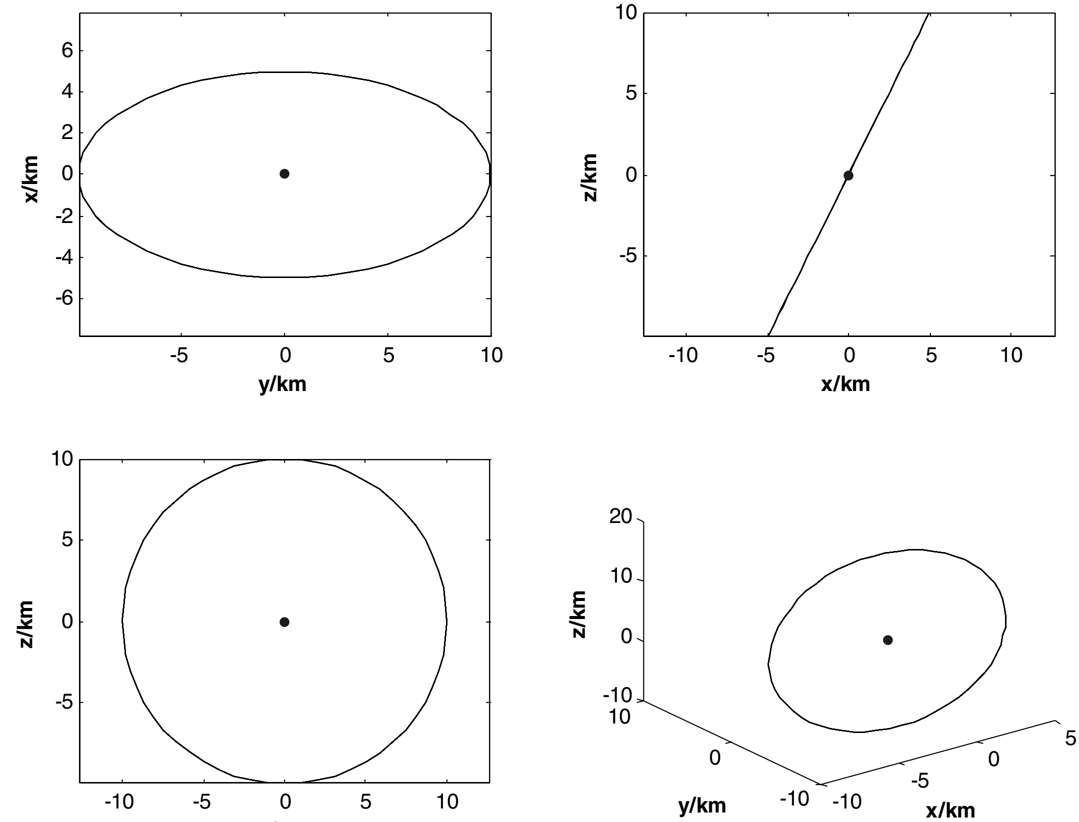

Fig. 3 Relative motions in the configure space with GPRMC.

Such an impulsive maneuver parameter optimization problem with the equality constraint can be solved using Lagrange multipliers. Augmenting the cost functional with the equality constraint using the Lagrange multipliers λ yields

$$\Phi = \|\Delta \mathbf{v}\|^2 + \lambda \Delta E \tag{18}$$

The necessary and sufficient conditions for the existence of minima are

$$\begin{cases} \frac{\partial \Phi}{\partial (\Delta \mathbf{v})} = 0 \\ \frac{\partial \Phi}{\partial \theta_f} = 0 \\ \frac{\partial \Phi}{\partial \lambda} = 0 \end{cases} \tag{19}$$

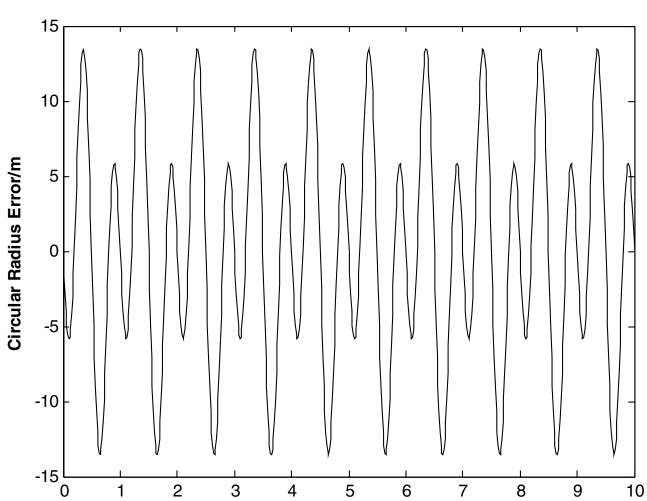

Fig. 4 Projected circular radius error between results of Hill's and nonlinear dynamic equations.

$$\frac{\partial^2 \Phi}{\partial (\Delta \mathbf{v})^2} > 0 \tag{20}$$

Equation (19) constitutes a system of five algebraic equations for the five variables Δv_x, Δv_y, Δv_z, θ_f, and λ.

Solving those equations, the corresponding optimal velocity corrections and the impulsive maneuver position are

$$\theta_f = 0, \qquad \Delta v_x = 0, \qquad \Delta v_z = 0 \tag{21}$$

$$\Delta v_y = -\sqrt{\frac{\mu(1+e_f)}{(a_l + \Delta a)(1-e_f)}} + \sqrt{\frac{\mu a_l(1+e_f) - \mu(1-e_f)\Delta a}{a_l(a_l+\Delta a)(1-e_f)}} \tag{22}$$

Assuming Δa is much smaller than a_l, Eq. (22) can be linearized to give

$$\Delta v_y = -\frac{n_l}{2}\sqrt{\frac{1-e_f}{1+e_f}}\Delta a \tag{23}$$

Equations (21) and (23) reassure the well-known fact that an optimal semimajor axis correction maneuver should be performed at periapsis.

B. Formation Keeping for Mission Requirement

The real formation-flying mission has some requirement for the formation array. We assume that that formation array can be described by a constraint equation

$$G(x, y, z, t) = 0 \tag{24}$$

Such a circular formation array is

$$x^2 + y^2 + z^2 = R^2 \tag{25}$$

And the projected circular formation array is

$$y^2 + z^2 = \rho^2 \tag{26}$$

When the formation mission constraint cannot be satisfied, we need the follower spacecraft to perform a maneuver to keep the formation array. To use Lagrange multipliers, the constraint Eq. (25) can be transformed to

$$\frac{\partial G}{\partial x}\dot{x} + \frac{\partial G}{\partial y}\dot{y} + \frac{\partial G}{\partial z}\dot{z} = 0 \tag{27}$$

In this case, the impulsive maneuver parameter optimization problems can be stated as follows:

Find an optimal impulsive maneuver $\Delta\mathbf{v}$, satisfying

$$\begin{gathered}\Delta v = \min_{\Delta v}\|\Delta\mathbf{v}\|^2 \\ \text{s.t. } \frac{\partial G}{\partial x}(\dot{x} + \Delta v_x) + \frac{\partial G}{\partial y}(\dot{y} + \Delta v_y) + \frac{\partial G}{\partial z}(\dot{z} + \Delta v_z) = 0\end{gathered} \tag{28}$$

Augmenting the cost functional with the equality constraint using the Lagrange multipliers λ yields

$$\Phi = \|\Delta\mathbf{v}\|^2 + \lambda\left[\frac{\partial G}{\partial x}(\dot{x} + \Delta v_x) + \frac{\partial G}{\partial y}(\dot{y} + \Delta v_y) + \frac{\partial G}{\partial z}(\dot{z} + \Delta v_z)\right] \tag{29}$$

The necessary and sufficient conditions for the existence of minima are

$$\left|\begin{array}{l}\frac{\partial\Phi}{\partial(\Delta\mathbf{v})} = 0 \\ \frac{\partial\Phi}{\partial\lambda} = 0\end{array}\right. \tag{30}$$

$$\frac{\partial^2\Phi}{\partial(\Delta\mathbf{v})^2} > 0 \tag{31}$$

Equation (30) constitutes a system of four algebraic equations for the four variables Δv_x, Δv_y, Δv_z, and λ. Solving Eq. (30), the corresponding optimal velocity corrections for mission requirement can be obtained.

Example 2: Considering the formation mission requires a circular formation.

When the circular formation cannot be satisfied, the optimal velocity corrections are

$$\Delta v_x = -\frac{x(x\dot{x} + y\dot{y} + z\dot{z})}{R^2} \qquad \Delta v_y = -\frac{y(x\dot{x} + y\dot{y} + z\dot{z})}{R^2} \qquad \Delta v_z = -\frac{z(x\dot{x} + y\dot{y} + z\dot{z})}{R^2} \tag{32}$$

V. Conclusions

A method to design the formation for spacecraft formation flying considering nonlinearity and eccentricity was presented, which employed generalized periodic relative motion to correct the initial conditions derived from Hill's equations. This approach is attractive for two reasons. First, the designed formation is bounded and periodic, and the secular drift is avoided in relative motion. Second, it adequately uses the previous research results which derived from Hill's equations and could use the intuitive methods to design formations for various formation-flying missions.

An impulsive formation-keeping strategy also was presented for keeping the periodic relative motion between formation spacecraft and satisfying the formation mission requirement. A single impulse was used to correct the initial errors and reduce the disturbing forces effects which destroyed the GPRMC. Another single impulse was used to keep the nominal formation and satisfy the mission requirement. If two impulses were used one after another, the all-type offset of the relative position and velocity vectors can be corrected.

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